Modification of the fluctuation dynamics of ultra-thin wetting films (Supplemental material)

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In these notes, we characterize the thermocapillary flow in a liquid layer of nanometric thickness. The thin film equation is derived in Section I in the lubrication approximation, and the stationary solution is given in the limit where the thermocapillary stress is balanced by van der Waals forces. In Section II, we show that the fluctuations of the free interface are controlled by both the Laplace pressure and intermolecular forces, whereas the thermocapillary contribution is irrelevant. We then derive the spectral density, which is compared in Section III with the more general expression obtained by solving the Stokes equation, without lubrication approximation.

I. THIN FILM EQUATION

I.1. Derivation of the equation

The evolution equation for the film thickness is derived in the lubrication approximation. The liquid is incompressible with viscosity η. We assume that the film thickness $h(\mathbf{r},t)$ is much smaller than the lateral extension of the liquid domain. The film is bounded below by a solid surface which coincides with the horizontal plane $z=0$, the z-axis being oriented upward. Throughout these notes, we use uppercase letters to denote 3D vectors whereas lowercases letters corresponds to their horizontal projection: for instance, the position vector is $\mathbf{R}=(\mathbf{r},z)$, with $\mathbf{r}=(x,y)$. At leading order, the Navier-Stokes equation reads

$$\eta\partial_z^2\mathbf{v}=\boldsymbol{\nabla}p\ , \tag{1}$$

with $\boldsymbol{\nabla}=(\partial_x,\partial_y)$. The solution for the velocity field is then straightforward: $\mathbf{v}=\mathbf{a}z+\mathbf{b}+z^2/(2\eta)\boldsymbol{\nabla}p$. The two constants $\mathbf{a}$ and $\mathbf{b}$ are set by the no-slip boundary condition $\mathbf{v}(\mathbf{r},0)=\mathbf{0}$ at the substrate ($z=0$), and the stress continuity condition $\eta\partial_z\mathbf{v}=\boldsymbol{\nabla}\gamma$ at the free interface ($z=h(\mathbf{r},t)$).

The temperature rise is assumed to be small so that we neglect the variations of the physical properties of the liquid. Indeed, a temperature rise of a few kelvins above the room temperature $T_0\approx 300$ K leads to variations that are of the order of 1% of the equilibrium values. We only account for the temperature dependance of the surface tension in the thermocapillary stress at the interface, since the latter actually induces the flow in the liquid layer. The surface tension γ is usually a decreasing fonction of the temperature. For small variations with respect to the unperturbed temperature T_0, one can assume a linear relation $\gamma(T)=\gamma_0-\gamma_\theta(T-T_0)$, with $\gamma_0=\gamma(T_0)$ and $\gamma_\theta=|\partial\gamma/\partial T|$. Defining $\theta(\mathbf{r})=T(\mathbf{r})-T_0$, one thus obtains

$$\mathbf{v}(\mathbf{r},z)=\frac{1}{2\eta}\left(z^2-2zh\right)\boldsymbol{\nabla}p-\frac{\gamma_\theta}{\eta}z\boldsymbol{\nabla}\theta\ . \tag{2}$$

Next, we enforce the incompressibility condition in order to relate the time evolution of the film thickness to the velocity field, namely

$$\partial_t h=-\boldsymbol{\nabla}\cdot\left(\int_0^h\mathbf{v}(\mathbf{r},z)\mathrm{d}z\right)=\boldsymbol{\nabla}\cdot\left(\frac{h^3}{3\eta}\boldsymbol{\nabla}p+\frac{h^2}{2\eta}\gamma_\theta\boldsymbol{\nabla}\theta\right)\ . \tag{3}$$

The pressure is finally expressed thanks to the balance of normal stresses at the interface. For films with a thickness of a few tens of nanometers, the contribution from gravity can be neglected and the stress balance condition reads

$$p_0-p=-\gamma_0\boldsymbol{\nabla}\cdot\mathbf{n}+\Pi(h)\ , \tag{4}$$

with $\mathbf{n}$ the local unit vector normal to the interface. The disjoining pressure is related to intermolecular forces, $\Pi(h)=-\Phi'(h)$, with $\Phi(h)=-A/(12\pi h^2)$ the non-retarded van der Waals interaction potential (per unit area). Note

that the Hamaker constant A is negative for repulsive van der Waals forces. In the small gradient approximation, the curvature moreover simplifies to $\boldsymbol{\nabla}\cdot\mathbf{n}\simeq-\nabla^2 h$. The thin film equation finally reads

$$\partial_t h=\boldsymbol{\nabla}\cdot\left(\frac{h^3}{3\eta}\boldsymbol{\nabla}\left[\phi'(h)-\gamma_0\nabla^2 h\right]+\frac{h^2}{2\eta}\gamma_\theta\boldsymbol{\nabla}\theta\right)\ . \tag{5}$$

I.2. Stationary solution

If the heating is axisymmetric, the temperature depends only on the radial distance $r=\sqrt{x^2+y^2}$. The stationary film profile is then solution of the equation

$$\partial_r\left(rh^3\partial_r\left[\frac{A}{6\pi h^3}-\gamma_0\nabla^2 h\right]+\frac{3}{2}\gamma_\theta r h^2\partial_r\theta\right)=0\ . \tag{6}$$

This equation states that the thinning of the film, due to the thermocapillary flow, is balanced by both the Laplace pressure and intermolecular forces. When the thickness of the film becomes smaller than $h_c=\left(|A|L^2/(6\pi\gamma_0)\right)^{1/4}$, with L the lateral scale on which thinning occurs, curvature effects can actually be neglected. As explained in the main body of the paper, L is in the millimeter range. Taking $|A|\approx10^{-19}$ J, $L\approx10^{-3}$ m and $\gamma_0\approx10^{-2}$ N·m^{-1}, one gets $h_c\approx1$ μm. For $h<h_c$, the term $\gamma_0\nabla^2 h$ can be disregarded and Eq. (6) can be integrated to give

$$\frac{A}{2\pi}\frac{h'}{h^3}=\frac{3}{2}\gamma_\theta\theta'\ , \tag{7}$$

where we have enforced the boundary conditions for the deformation profile, $h(r\to\infty)=h_\infty$ and $\partial_r h|_\infty=0$, and for the temperature, $\theta(r\to\infty)=0$ and $\partial_r\theta|_\infty=0$.

Integrating once more between 0 and r, one arrives at

$$\frac{1}{h_0^2}-\frac{1}{h(r)^2}=-\frac{6\pi\gamma_\theta}{A}\left[\theta(0)-\theta(r)\right]\ , \tag{8}$$

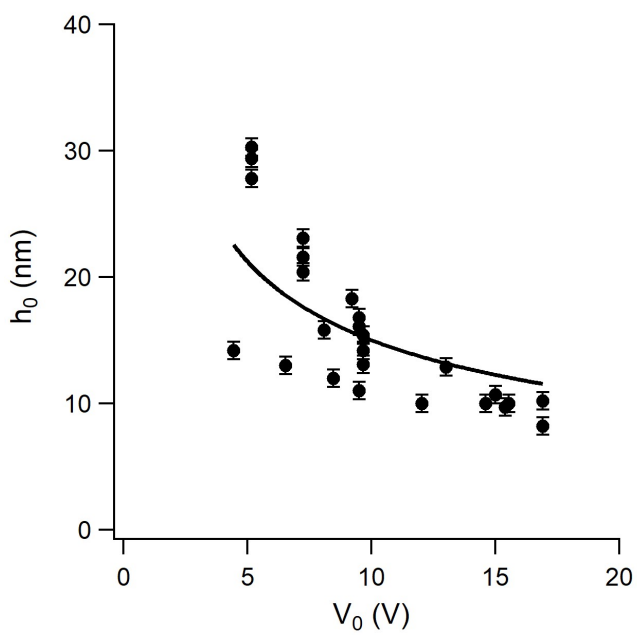

FIG. 1. Evolution of the stationary thickness as a function of the voltage amplitude V_0 (in volts) delivered by the two quadrants of the photodiode, which is proportional to the light intensity on the quadrants and hence to the temperature increase. The results are consistent with the theoretical prediction Eq. (9) assuming that the maximum temperature rise is proportional to the laser intensity (the full line corresponds to $V_0^{-1/2}$). The dispersion of the data is presumably due to the presence of adsorbed contaminants at the interface.

with $h_0 = h(0)$. The thickness at the center is finally obtained by taking the limit $r \to \infty$, together with the assumption $h_0 \ll h_\infty = h(r \to \infty)$, so that[1]

$$h_0 = \sqrt{\frac{|A|}{6\pi\gamma_\theta\theta(0)}} \ . \tag{9}$$

For $|A| \approx 10^{-19}$ J, $\gamma_\theta \approx 10^{-4}$ N·m^{-1}·K^{-1} and $\theta(0) \approx 1$ K, we predict a stationary thickness $h_0 \approx 10$ nm, which is indeed much smaller than the initial thickness $h_\infty \approx 1$ μm.

II. STOCHASTIC THIN FILM EQUATION AND FLUCTUATION SPECTRUM

II.1 Linearization of the thin film equation

For small deviations with respect to the stationary height h_0, one can set $h = h_0 + \delta h$ with $|\delta h| \ll h_0$. Eq. (5) can then be linearized and one gets at lowest order

$$\partial_t \delta h = \frac{h_0}{3\eta}\boldsymbol{\nabla}\cdot\left[h_0^2\Phi_0''\boldsymbol{\nabla}\delta h - h_0^2\gamma_0\boldsymbol{\nabla}\left(\nabla^2\delta h\right) + 3\gamma_\theta\delta h\boldsymbol{\nabla}\theta\right] \ , \tag{10}$$

with $\Phi_0'' = \Phi''(h_0)$. The issue is then to evaluate which contributions are relevant regarding the fluctuation spectrum. To estimate the relative weight of each contribution in the square brackets, we first note that θ and δh vary on different length scales. Indeed, the temperature gradient is set by the experimental set-up, $|\boldsymbol{\nabla}\theta| \approx \theta(0)/L$, whereas the deformation gradient involves an intrinsic length scale λ such that $|\boldsymbol{\nabla}\delta h| \approx \delta h/\lambda$. Equating the contribution from Laplace pressure and van der Waals forces, one gets

$$\lambda = \left(\frac{2\pi h_0^4\gamma_0}{|A|}\right)^{1/2} \ . \tag{11}$$

Taking the same numerical values as above, we find that λ is in the micrometer range. We then have

$$\frac{|\gamma_\theta\delta h\boldsymbol{\nabla}\theta|}{|h_0^2\Phi_0''\boldsymbol{\nabla}\delta h|} \approx \frac{\lambda}{L} \approx 10^{-3} \ , \tag{12}$$

so that the thermocapillary contribution can be neglected when focusing on the fluctuations of the film thickness.

II.2. Stochastic thin film equation

According to the prior discussion, we now rewrite the thin film Eq. (10) where the thermocapillary contribution is neglected. Although the stationary height is set by the *nonequilibrium* Marangoni flow, the thin film fluctuations follow an *equilibrium* distribution. We then adapt the original analysis of Mecke and Rauscher [On thermal fluctuations in thin film flow, J. Phys.: Condens. Matter **17**, S3515 (2005)] and include a random noise term in order to account for fluctuations. The stochastic version of the thin film equation then reads

$$\partial_t \delta h = \frac{h_0^3}{3\eta}\left[\phi''(h_0)\nabla^2\delta h - \gamma_0\nabla^4\delta h\right] + \boldsymbol{\nabla}\cdot\mathbf{f} \ , \tag{13}$$

where the correlations of the Gaussian white noise $\mathbf{f}$ are given by

$$\langle f_i(\mathbf{r},t)\rangle = 0 \ , \quad \text{and} \quad \langle f_i(\mathbf{r},t)f_j(\mathbf{r}',t')\rangle = \left(\frac{2k_BT_0h_0^3}{3\eta}\right)\delta_{ij}\delta(t-t')\delta(\mathbf{r}-\mathbf{r}') \ . \tag{14}$$

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[1] As a matter of fact, the limit $r \to \infty$ has to be considered with caution since capillary effects are not negligible in the unperturbed region. Still, the thickness at the center Eq. (9) remains correctly given by the balance between thermocapillary stresses and van der Waals forces.

Switching to time-Fourier representation

$$\delta\tilde{h}_{\mathbf{q},\omega} = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi} \int \mathrm{d}^2\mathbf{r}\, e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \delta h(\mathbf{r},t) \ , \tag{15}$$

the solution of Eq. (13) is then straigthforward

$$\delta\tilde{h}_{\mathbf{q},\omega} = \frac{i\mathbf{q}\cdot\tilde{\mathbf{f}}(\mathbf{q},\omega)}{\omega_q - i\omega} \ , \tag{16}$$

where we define $\omega_q = \Gamma(q,h_0)\epsilon_q$, with $\Gamma(q,h_0) = q^2 h_0^3/(3\eta)$ and $\epsilon_q = \gamma_0 q^2 - A/(2\pi h_0^4)$. Regarding the experiments, one is interested in the spectral density $P(\mathbf{q},\omega)$ which is defined as the Fourier transform of the height correlation function

$$\left\langle \delta\tilde{h}_{\mathbf{q},\omega}\delta\tilde{h}_{\mathbf{q}',\omega'} \right\rangle = P(\mathbf{q},\omega)\delta^2(\mathbf{q}+\mathbf{q}')\delta(\omega+\omega') \ . \tag{17}$$

Here, we introduce the shorthand notation $\delta^2(\mathbf{q}+\mathbf{q}') = 2\pi^2\delta(\mathbf{q}+\mathbf{q}')$. One finally obtains the spectral density

$$P(\mathbf{q},\omega) = \left\langle \left|\delta\tilde{h}_{\mathbf{q},\omega}\right|^2 \right\rangle = \frac{k_B T_0}{\pi} \frac{\Gamma(q,h_0)}{\omega^2+\omega_q^2} \ . \tag{18}$$

III. THIN FILM *VS.* STOKES RELAXATION

So far, we have assumed that the lubrication approximation is appropriate. We still have to check whether the thin film regime is suitable to describe the normal fluctuations of the interface. To this aim, we solve the full Stokes problem without the lubrication approximation and derive the fluctuation spectrum of the interface. We then discuss the results in the regime that is experimentally relevant.

III.1. Dispersion relation

Assuming Re $\ll 1$, the velocity and pressure fields are solution of the incompressible Stokes equations

$$\eta\nabla^2\mathbf{v} - \boldsymbol{\nabla} p = 0 \ , \quad \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \ , \tag{19}$$

together with the no-slip condition $\mathbf{v}(\mathbf{r},0) = \mathbf{0}$ at the substrate ($z=0$). At the free interface ($z = h(\mathbf{r})$), we have to enforce the stress balance condition

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)\cdot\hat{\mathbf{n}} = \left[-\gamma\left(\boldsymbol{\nabla}\cdot\hat{\mathbf{n}}\right) + \Pi(h)\right]\hat{\mathbf{n}} \ , \tag{20}$$

where the components of the stress tensor $\boldsymbol{\sigma}$ in the liquid phase are $\sigma_{ij} = -p\delta_{ij} + \eta\left(\partial_i v_j + \partial_j v_i\right)$. The stress tensor in the gas phase is simply $\boldsymbol{\sigma}_0 = -p_0\boldsymbol{I}$, with $\boldsymbol{I}$ the identity tensor and p_0 the pressure. Finally, the time-evolution equation for the interface is obtained from the kinematic condition

$$\partial_t h = v_z(\mathbf{r},h,t) \ . \tag{21}$$

The deformation of the interface is then written as $h = h_0 + \delta h$, with $|\delta h| \ll h_0$. For the analysis that follows, it is convenient to define the Fourier representation

$$\delta\tilde{h}_{\mathbf{q}}(z,t) = \int \mathrm{d}^2\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \delta h(\mathbf{r},z,t) \ , \tag{22}$$

with $\mathbf{r} = (x,y)$ and $\mathbf{q} = (q_x,q_y)$. As shown in the Appendix, the time-evolution equation for the interface can then be written as

$$\partial_t\delta\tilde{h}_{\mathbf{q}} = -\gamma_q\delta\tilde{h}_{\mathbf{q}} \ . \tag{23}$$

Here, the damping rate γ_q is expressed quite generally as $\gamma_q = \Gamma_{St}(q,h_0)\epsilon_q$, with $\epsilon_q = \gamma_0 q^2 - A/(2\pi h_0^4)$. The dissipation kernel is $\Gamma_{St}(q,h_0) = \Gamma_0(q)F(qh_0)$, with $\Gamma_0(q) = 1/(2\eta q)$ the Oseen kernel (pertaining to a semi-infinite liquid layer). Hydrodynamic interactions with the wall can be factorized into a single function F defined as

$$F(u) = \frac{1 - 4ue^{-2u} - e^{-4u}}{1 + 2(1+2u^2)e^{-2u} + e^{-4u}} \ . \tag{24}$$

It behaves asymptotically as $F(u) \sim 2u^3/3$ when $u \to 0$, and $F(u) \to 1$ when $u \to \infty$.

III.2. Fluctuation spectrum

We account for the fluctuations by adding a random term to the relaxation Eq. (23), that now reads

$$\partial_t \delta\tilde{h}_{\mathbf{q}} = -\gamma_q \delta\tilde{h}_{\mathbf{q}} + \mathcal{N}_{\mathbf{q}} \ . \tag{25}$$

The white noise $\mathcal{N}_{\mathbf{q}}$ follows a Gaussian statistics, with zero mean value $\langle \mathcal{N}_{\mathbf{q}}(t) \rangle = 0$ and correlations given by the fluctuation-dissipation theorem

$$\langle \mathcal{N}_{\mathbf{q}}(t) \mathcal{N}_{\mathbf{q}'}(t') \rangle = 2 k_B T_0 \Gamma_{St}(q, h_0) \delta^2(\mathbf{q} + \mathbf{q}') \ . \tag{26}$$

We then switch to the time-Fourier representation so that the solution of the Langevin Eq. (25) is straightforward

$$\delta\tilde{h}_{\mathbf{q},\omega} = \frac{\mathcal{N}_{\mathbf{q},\omega}}{\gamma_q - i\omega} \ , \quad \text{with} \quad \langle \mathcal{N}_{\mathbf{q},\omega} \mathcal{N}_{\mathbf{q}',\omega'}(t') \rangle = \frac{k_B T}{\pi} \Gamma_{St}(q, h_0) \delta^2(\mathbf{q} + \mathbf{q}') \delta(\omega + \omega') \ . \tag{27}$$

The final result is then

$$P_{St}(\mathbf{q}, \omega) = \frac{k_B T}{\pi} \frac{\Gamma_{St}(q, h_0)}{\omega^2 + \gamma_q^2} \ . \tag{28}$$

Note that Eq. (28) shares many similarities with the thin-film result Eq. (18). They only differ in the expression of the dissipation kernel Γ.

III.3. Comparison between the lubrication approximation and the full Stokes solution

To start the discussion, let us compare the relaxation rates of the two models. When solving the full Stokes problem, we find that the relaxation rate is given by

$$\gamma_q = \gamma_q^{(0)} F(q h_0) \ , \quad \text{with} \quad \gamma_q^{(0)} = \frac{1}{2\tau_c} \left(\frac{1 + (\lambda q)^2}{\lambda q} \right) \ . \tag{29}$$

This relation can be interpreted as follows. Here, $\gamma_q^{(0)}$ is the relaxation rate in the thick film limit ($h_0 \to \infty$), with $\tau_c = \lambda\eta/\gamma_0$ the capillary time. For finite h_0, the relaxation rate is modified since viscous dissipation increases (no slip condition at the bounding wall). The asymptotic limits of Eq. (29) are $\gamma_q \sim \gamma_q^{(0)}$ when $q h_0 \to \infty$, and

$$\gamma_q = \frac{1}{\tau_0} (\lambda q)^2 \left[1 + (\lambda q)^2 \right] \ , \tag{30}$$

when $q h_0 \to 0$, where we define $\tau_0 = 3\eta\lambda^4/(\gamma_0 h_0^3)$.

On the other hand, in the thin film limit, one actually gets the same expression Eq. (30) for the relaxation rate ω_q. This is actually no big surprise since the long wave-length approximation $q h_0 \ll 1$ is actually underlying the thin film approximation. To see which regime is experimentally relevant, we can evaluate the relevant length and time scales. We take the numerical values $h_0 \approx 10$ nm, $|A| = 10^{-19}$ J, $\gamma_0 = 10^{-2}$ N·m^{-1}, and $\eta = 10^{-2}$ Pa·s.

- Length scales: the PSD measured in the experiment involves an integral of $P(\mathbf{q}, \omega)$ with a weight function that is maximum for wavevectors $q \sim 1/R$, with $R \approx 10^{-6}$ m the beam size. With $h_0 \approx 10^{-8}$ m that is 2 orders of magnitude smaller than R, the lubrication approximation is fully justified: $q h_0 \sim h_0/R \ll 1$.

- Time scales: $\tau_c = \eta\lambda/\gamma_0 \simeq 10^{-7}$ s, which is several orders of magnitude smaller than $\tau_0 \simeq 10^{-4}$ s.

As a consequence, the thin regime limit is therefore a very good approximation in the low frequency limit. The full solution of the Stokes equation becomes relevant when the surface tension is sufficiently small (*e.g.*, close to a critical point), or when the thickness of the film increases (one then recovers the results for the semi-infinite layer).

Regarding the correlation function, the complete solution Eq. (28) is always valid, whatever the values of the parameters. It can be used to fit the data from the thick layer regime down to the very thin layer regime. Still, since the system is highly confined, the thin film expression (18) is a very good approximation.

APPENDIX

In this appendix, we derive the solution of the Stokes problem. We define the cylindrical basis $(\hat{\mathbf{q}}, \hat{\mathbf{t}}, \hat{\mathbf{n}})$ in Fourier space

$$\hat{\mathbf{q}} = \frac{q_x}{q}\mathbf{e}_x + \frac{q_y}{q}\mathbf{e}_y \ , \quad \hat{\mathbf{t}} = -\frac{q_y}{q}\mathbf{e}_x + \frac{q_x}{q}\mathbf{e}_y \ , \quad \text{and} \quad \hat{\mathbf{n}} = \mathbf{e}_z \ . \tag{31}$$

In this system of coordinates, the velocity is decomposed in longitudinal, transverse and normal components: $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}_{\parallel} + \tilde{v}_z\hat{\mathbf{n}} = \tilde{v}_l\hat{\mathbf{q}} + \tilde{v}_t\hat{\mathbf{t}} + \tilde{v}_z\hat{\mathbf{n}}$. The Stokes Eq. (19) then reads

$$\partial_z^2\tilde{v}_l - q^2\tilde{v}_l = \frac{iq}{\eta}\tilde{p} \ , \tag{32a}$$

$$\partial_z^2\tilde{v}_t - q^2\tilde{v}_t = 0 \ , \tag{32b}$$

$$\partial_z^2\tilde{v}_z - q^2\tilde{v}_z = \frac{1}{\eta}\partial_z\tilde{p} \ , \tag{32c}$$

$$iq\tilde{v}_l + \partial_z\tilde{v}_z = 0 \ . \tag{32d}$$

The pressure can be related to the normal component of the velocity through: $\tilde{p} = \eta\left(q^{-2}\partial_z^3\tilde{v}_z - \partial_z\tilde{v}_z\right)$. From Eqs. (32a), (32c) and (32d), we can deduce a single equation for the normal component

$$\partial_z^4\tilde{v}_z - 2q^2\partial_z^2\tilde{v}_z + q^4\tilde{v}_z = 0 \ . \tag{33}$$

Regarding the boundary conditions, we assume the no-slip condition $\mathbf{v}(\mathbf{r}, 0) = \mathbf{0}$ at the substrate, so that in Fourier representation

$$\tilde{v}_z(q, 0) = 0 \ , \tag{34a}$$

$$\tilde{v}_t(q, 0) = 0 \ , \tag{34b}$$

$$\partial_z\tilde{v}_z(q, 0) = 0 \ . \tag{34c}$$

The latter follows from the incompressibility condition Eq. (32d). At the free interface, we have to enforce the stress balance condition Eq. (20) at $z = h(\mathbf{r}, t) = h_0 + \delta h(\mathbf{r}, t)$, where we assume $|\delta h| \ll h_0$. We then develop the disjoining pressure at linear order: $\Pi(h) = \Pi(h_0) + \Pi'(h_0)\delta h + \mathcal{O}(\delta h^2)$. The boundary condition Eq. (20) is then projected on the perpendicular and parallel directions.

- $\mathbf{e}_i \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) \cdot \mathbf{e}_z = 0$ (with $i = x, y$), so that in Fourier representation

  $$\eta\left(\partial_z\tilde{\mathbf{v}}_{\parallel} + i\mathbf{q}\tilde{v}_z\right)\Big|_{z=h_0} = \mathbf{0} \ .$$

  We then obtain in the transverse direction

  $$\partial_z\tilde{v}_t\big|_{z=h_0} = 0 \ . \tag{35}$$

  In the longitudinal direction, the BC leads to $\left(\partial_z\tilde{v}_l + iq\tilde{v}_z\right)\big|_{z=h_0} = 0$, which can be rewritten thanks to the continuity Eq. (32d) as

  $$\left(\partial_z^2\tilde{v}_z + q^2\tilde{v}_z\right)\big|_{z=h_0} = 0 \ . \tag{36}$$

- $\mathbf{e}_z \cdot \sigma \cdot \mathbf{e}_z = [\Pi(h_0) + \Pi'(h_0)\delta h] + \gamma\boldsymbol{\nabla}^2\delta h$. Without loss of generality one can set $p_0 = \Pi(h_0)$, so that we get

  $$\left(2\eta\partial_z\tilde{v}_z - \tilde{p}\right)\big|_{z=h_0} = -\epsilon_q\delta\tilde{h}_{\mathbf{q}} \ ,$$

  with $\epsilon_q = \gamma_0 q^2 - \Pi'(h_0)$. Substituting the pressure in the latter equation, we finally get

  $$\left(\partial_z^3\tilde{v}_z - 3q^2\partial_z\tilde{v}_z\right)\big|_{z=h_0} = \frac{q^2\epsilon_q}{\eta}\delta\tilde{h}_{\mathbf{q}} \ . \tag{37}$$

We first focus on the transverse component: it satisfies Eq. (32b), whose solution is a linear combination of $\sinh(qz)$ and $\cosh(qz)$. Enforcing the BCs (34b) and (35) then leads to the trivial solution

$$\tilde{v}_t(q,z) = 0 \ . \tag{38}$$

The normal component of the velocity is solution of Eq. (33), whose solution is straightforward

$$\tilde{v}_z(q,z) = Ae^{-qz} + Bqze^{-qz} + Ce^{qz} + Dqze^{qz} \ . \tag{39}$$

The integration constants are obtained by enforcing the BCs (34a), (34c), (36) and (37), *i.e.* in matrix notation

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ e^{-u} & (-1+u)e^{-u} & e^{u} & (1+u)e^{u} \\ e^{-u} & ue^{-u} & -e^{u} & -ue^{u} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma_0 \epsilon_q \delta\tilde{h}_q \end{pmatrix} ,$$

with $u = qh_0$ and $\Gamma_0 = 1/(2\eta q)$. This system is readily inverted and we get

$$A = \left[\frac{1+u+(1-u)e^{-2u}}{1+2(1+2u^2)e^{-2u}+e^{-4u}} \right] e^{-u}\Gamma_0\epsilon_q\delta\tilde{h}_{\mathbf{q}} \ , \tag{40a}$$

$$B = \left[\frac{1+2u+e^{-2u}}{1+2(1+2u^2)e^{-2u}+e^{-4u}} \right] e^{-u}\Gamma_0\epsilon_q\delta\tilde{h}_{\mathbf{q}} \ , \tag{40b}$$

$$C = -\left[\frac{1+u+(1-u)e^{-2u}}{1+2(1+2u^2)e^{-2u}+e^{-4u}} \right] e^{-u}\Gamma_0\epsilon_q\delta\tilde{h}_{\mathbf{q}} \ , \tag{40c}$$

$$D = \left[\frac{1+(1-2u)e^{-2u}}{1+2(1+2u^2)e^{-2u}+e^{-4u}} \right] e^{-u}\Gamma_0\epsilon_q\delta\tilde{h}_{\mathbf{q}} \ . \tag{40d}$$

The solution of the Stokes problem is then given by Eq. (39) with the integration constants obtained in Eq. (40).